Math 55 Discussion problems 23 Feb

1. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.
2. Prove that for every positive integer n, $\Sigma_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$.
3. Suppose that a and b are real numbers with $0 < b < a$. Prove that if n is a positive integer, then $a^n - b^n \leq na^{n-1}(a-b)$.
4. Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.
5. Prove that if $A_1, A_2, ..., A_n$ and B are sets, then $(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B$.
6. Show that n lines separate the plane into $\frac{n^2+n+2}{2}$ regions if no two of these lines are parallel and no three pass through a common point.
7. Suppose that P is a simple polygon with vertices $v_1, v_2, ..., v_n$ listed so that consecutive vertices are connected by an edge, and v_1 and v_n are connected by an edge. A vertex v_i is called an ear if the line segment connecting the two vertices adjacent to v_i is an interior diagonal of the simple polygon. Two ears v_i and v_j are called nonoverlapping if the interiors of the triangles with vertices v_i and its two adjacent vertices and v_j and its two adjacent vertices do not intersect. Prove that every simple polygon with at least four vertices has at least two nonoverlapping ears.
8. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 18$.
    (a) Show that the statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of a proof by strong induction that P(n) is true for all integers $n \geq 18$.
    (b) What is the inductive hypothesis of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$?
    (c) What do you need to prove in the inductive step of a proof that $P(n)$ is true for all integers $n \geq 18$?
    (d) Complete the inductive step for $k \geq 21$.
    (e) Explain why these steps show that $P(n)$ is true for all integers $n \geq 18$.